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OPTIMAL GROWTH UNDER ENDOGENOUS DEPRECIATION, CAPITAL UTILIZATION AND MAINTENANCE COSTS

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This paper analyzes the equilibrium dynamics of an optimal growth model with endogenous depreciation, variable capital utilization, and expenditures on the maintenance of physical capital. Maintenance reduces the depreciation of capital, investment is subject to adjustment costs, and the degree of capital utilization affects the activity of maintainance. We establish a set of sufficient conditions for the existence and uniqueness of a steady state equilibrium. We define a "delta golden rule" consistent with the proposed economic environment and we analyze the dynamic efficiency of this economy. Finally, the steady state is found to be locally saddle-path stable.

Key words: Maintenance, depreciation, capital utilization, optimal growth.

(*JEL* O40, E22, D90)

1. Introduction

Most analyses of aggregate economic activity take depreciation as exogenously given and ignore that equipment and structures have to be maintained and repaired. Moreover, an important margin along which a firm can adjust these activities has to do with the fraction of the installed capital stock being used. In this paper we develop a neoclassical growth framework that incorporates the endogenous determination of these variables. To this end, we augment the optimal growth model of saving and investment under adjustment costs introduced by Abel and Blanchard (1983), with a maintenance technology that acts as a

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substitute for investment and depends upon both the intensity with which physical capital is utilized and its depreciation rate.

The exogenous nature of physical capital depreciation can be somewhat justified by considering a class of *putty-putty* models of production. However, it is well understood that this particular view of the capital accumulation process is quite restrictive. From the theoretical side, the assumption of exponential depreciation dramatically reduces the possible dynamics that an optimal growth model can describe. This is particularly relevant for growth theory as well as for investment theory. From the empirical side, the assumption of a constant depreciation rate turns out to be more in conformity with accounting principles than with those of economic theory. This is particularly important with respect to the measurement of physical capital.

There is evidence that the activity of maintaining and repairing equipment and structures (cf. McGrattan and Schmitz, 1999) is both large relative to investment and a substitute for investment to some extent. Furthermore, Licandro and Puch (2000) show that incorporating expenditures on the maintenance and repair of physical capital into models of aggregate economic activity will change the quantitative answers to some key questions that have been addressed with these models.[1] What it is missing is an analytical framework to characterize the equilibrium dynamics of the joint determination of investment rates, depreciation rates and utilization rates. Here, and this is the contribution of the paper, we give a first step in that direction by incorporating to the neoclassical growth framework a class of maintenance activities that are related with the capital ageing process and the decay that results from its use.

Our model specification builds upon previous results in Escribá-Pérez and Ruiz-Tamarit (1996) and Ruiz-Tamarit (1995). These authors explore the endogenous determination of depreciation under *putty-putty* technologies in partial equilibrium. In doing so, they introduce a maintenance activity that allows a reduction in physical depreciation, which is positively related in turn with the intensity of use of capital under the depreciation-in use assumption.[2] We put these ideas to work into

[1] See also Collard and Kollintzas (2000).

[2] Different specifications of this hypothesis have been discussed in Epstein and Denny (1980), Bischoff and Kokkelenberg (1987), Motahar (1992) and Burnside and Eichenbaum (1996). See also Rumbos and Auernheimer (1997) and the references therein.

a general equilibrium framework.

The general equilibrium neoclassical growth model does not allow the separation of the saving decisions of households from the investment decisions of firms. By introducing either a two sector technology (cf. Uzawa, 1964) or installation costs (cf. Abel and Blanchard, 1983) it is possible to overcome the essentially passive role of investment in the model. These analyses, however, rely on constant depreciation rates and full capacity utilization. Here we incorporate a maintenance technology into the standard growth model with adjustment costs. This technology allows us to augment the model to include endogenous depreciation and capital underutilization. Thus, under the necessary assumptions to derive well-defined investment, depreciation, and utilization functions, we characterize the steady state equilibrium, the short-run dynamics and the stability properties of our model.

As a result of our technological assumptions, capital underutilization is optimal and the investment rate is determined simultaneously with the endogenous depreciation rate. Consequently, the equilibrium path is dynamically efficient. Furthermore, the long-run equilibrium of our economy is characterized by a unique optimal capital stock which is below that of the standard neoclassical growth model with adjustment costs. Consequently, our technological assumptions suggest that a non-optimal depreciation policy, that ignores maintenance costs and variable utilization, might lead the economy to an excess of installed capacity.

In addition, we focus on the analysis of the dynamic properties of optimal trajectories. Once we prove local stability we present a set of numerical computations. We shall illustrate below that the presence of a simple maintenance costs technology can reasonably reduce the rate of convergence to the steady-state path. It turns out that these values of the speed of convergence are more in conformity with those supported by the empirical evidence. Also, along the optimal trajectories consumption, capital and output are positively related but the investment rate, defined as the ratio of investment over capital, is inversely related with them. Depreciation, utilization and maintenance rates are also inversely related with capital accumulation along the convergence path.

These findings provide a framework for the analysis of comparative dynamics in general equilibrium with these features. The rest of the

paper is organized as follows. In Section 2 we introduce the model along with a discussion of the maintenance and adjustment costs technologies. In Section 3 we show existence and uniqueness of steady state equilibrium and we characterize optimal solutions. In Section 4 we present stability results. Section 5 concludes.

2. The model and preliminary considerations

The goal here is to formalize endogenous depreciation, capital utilization and maintenance costs in the simplest optimal growth economy. In addition, we retain the assumption that investment is subject to an adjustment costs technology. The reason is twofold. On the one hand, this allows us to keep as a benchmark the specification in Abel and Blanchard (1983). On the other hand, the presence of a simple maintenance costs technology can be immediately justified in an economy where adjustment costs generate a well-defined investment demand function. We now introduce a general model of optimal growth with these features.

The economy is populated by a continuum of identical infinitely-lived households or dynasties that grow at an exogenously given rate $n \geq 0$. Each household discounts the future at a constant positive rate $\beta > n$, and derives instantaneous utility from the consumption of an aggregate good, c_t, according to $U(c_t)$, which is an increasing and strictly concave C^2 mapping.

The technology is represented by a C^2 concave production function, $F(\tilde{K}, N)$, which is increasing and linearly homogeneous in effective capital, $\tilde{K} > 0$, and labor, $N > 0$. The fraction $0 < u < 1$ determines the intensity of use of the installed capital stock K, thus $\tilde{K} = K\,u$. N is also the population size. For simplicity of exposition we ignore the immediate extension to the case of exogenous technical progress of the labor augmenting type. Under the previous assumptions on $F(.)$,

$$y_t = f(k_t\,u_t), \qquad [1]$$

where y and k are *per capita* output and *per capita* capital stock, respectively. Function $f(x)$ is C^2, increasing and strictly concave for all $x > 0$, $\lim_{x \to 0^+} f(x) = 0$ and $f(.)$ satisfies the Inada conditions.

Production may be allocated to consumption, the production of new capital goods, installation activities and maintenance services. Associated to these purchases are the two key ingredients of the present

analysis, namely: *i)* investment is subject to an adjustment costs technology, and *ii)* maintenance and repair are subject to a maintenance costs technology.

Let us assume that adjustment costs, which are internal to the firm, are represented by a linearly homogeneous function $\Phi(I, K)$, increasing in gross investment, $I > 0$, and decreasing in the total capital stock, $K > 0$. Then $\Phi(I, K) = \phi(i)K$, where i is the rate of gross investment and $\phi(i)$ is assumed C^2, increasing and strictly convex for $i > 0$, with $\lim_{i \to 0+} \phi(i) = 0$ and $\lim_{i \to +\infty} \phi'(i) = +\infty$. Consequently, *per capita* adjustment costs are then written as $\phi(i)k$.

By assumption, maintenance costs are internal to the firm and can be used to preserve capital goods from depreciation and use. These maintenance costs are represented by a linearly homogeneous function $M(D, \tilde{K})$, decreasing in total depreciation $D > 0$ and increasing in effective capital. Consequently, $M(D, \tilde{K}) = m(\delta, u)K$, where $\delta > 0$ is the endogenous rate of depreciation. The function $m(\delta, u)$, the average maintenance costs, is assumed positive, C^2, convex and linearly homogeneous on $\delta > 0$ and $u \in]0, 1[$. Furthermore, we assume $m_\delta(\delta, u) < 0$, $m_u(\delta, u) > 0$, $m_{\delta\delta}(\delta, u) > 0$, $m_{uu}(\delta, u) > 0$ and $m_{\delta u}(\delta, u) < 0$ for $u \in]0, 1[$ and $\delta > 0$. The larger the utilization rate and the smaller the depreciation rate, the larger the maintenance costs of capital.[3]

The resource constraint is determined by the following equalities

$$c_t + (i_t + \phi(i_t) + m(\delta_t, u_t))\ k_t = f(k_t\ u_t) \qquad [2]$$

$$\dot{k}_t = (i_t - \delta_t - n)\ k_t, \qquad [3]$$

where $\dot{k}$ denotes the time derivative of *per capita* physical capital with respect to time.

In the present setting, every optimal solution may be decentralized as a competitive equilibrium. Thus, without loss of generality we shall confine our analysis to the planner's problem. The planner's optimization problem is to choose at each moment in time the rates of investment, depreciation and capital utilization so as to maximize the infinite stream of discounted instantaneous utilities, given the resource constraints [2] and [3], and the initial capital stock, k_0.

[3] An equivalent representation of the problem can be achieved by assuming that the depreciation rate is a function of the utilization rate and the rate of maintenance costs to capital.

DEFINITION 1. An optimal solution for this economy is a set of paths $\{c_t, i_t, \delta_t, u_t, k_t\}$, for t positive, which solve

$$\max \int_0^{\infty} U(c_t)\ e^{-(\beta-n)t}\ dt, \qquad [P]$$

subject to [2] and [3], $k_0 > 0$ given, and where all variables are assumed to be strictly positive and u_t strictly smaller than one.

DEFINITION 2. A steady-state equilibrium for this economy is an optimal solution $\{c_t, i_t, \delta_t, u_t, k_t\}$ to problem [P], such that k_t, c_t, i_t, δ_t and u_t remain constant.

It is readily shown that not only rates but also equilibrium *per capita* variables remain constant at steady state. Consequently, at a steady state the equilibrium levels grow at the rate n.

An interior optimal solution to problem [P] must satisfy, dropping time subscripts, the following first order equation system

$$\mu = U'(c)\ \ (1 + \phi'(i)) \qquad [4]$$

$$\mu = -U'(c)\ m_{\delta}(\delta, u) \qquad [5]$$

$$f'(k\,u) = m_u(\delta, u) \qquad [6]$$

$$\dot{\mu} = U'(c)\,(i + \phi(i) + m(\delta, u) - f'(k\,u)u) + \mu\,(\beta + \delta - i) \qquad [7]$$

$$\dot{k} = (i - \delta - n)k \qquad [8]$$

$$c + k\ \ (i + \phi(i) + m(\delta, u)) = f(k\ u) \qquad [9]$$

given $k_0 > 0$ and the corresponding transversality condition

$$\lim_{t\to\infty} \mu_t\ k_t\ e^{-(\beta-n)t} = 0. \qquad [10]$$

The multiplier μ represents the shadow price of an additional unit of installed capital. The term $1 + \phi'(i)$ is the marginal opportunity cost of gross investment. Then, equation [4] states that this marginal cost must be equal to the shadow price of capital. On the other hand, $-m_{\delta}(\delta, u)$ is the marginal saving in maintenance costs associated to an increase in the depreciation rate. An increase in δ reduces the capital stock and, hence, maintenance expenditures. So, equation [5] states that this marginal saving must be equal to the shadow price of capital. The term $m_u(\delta, u)$ is the marginal maintenance cost associated to an increase in the utilization rate. Equation [6] states that this marginal

cost must be equal to the marginal productivity of such an increase in the utilization rate, measured by the term $f'(k\ u)$.

From [4] to [6] and [9], we can write $c = c(k, \mu)$, $i = i(k, \mu)$, $\delta = \delta(k, \mu)$ and $u = u(k, \mu)$. Additionally, combining [4] to [7] and the assumption that $m(\delta, u)$ is linearly homogeneous, the dynamic system [7] and [8] can be written in the phase space as

$$\frac{\dot{\mu}}{\mu} = \beta - H\left(i(k, \mu)\right) \equiv \frac{\Gamma(k, \mu)}{\mu} \qquad [11]$$

$$\frac{\dot{k}}{k} = i(k, \mu) - \delta(k, \mu) - n \equiv \frac{\Pi(k, \mu)}{k}, \qquad [12]$$

where

$$H(i) = \frac{i\phi'(i) - \phi(i)}{1 + \phi'(i)}.$$

Indeed, $H(i)$ summarizes all the marginal effects on the return to capital. From the assumed properties of function $\phi(i)$, we can easily prove that $H(i) > 0$, $H'(i) > 0$ and $H(i) < i$, for all $i > 0$. Moreover, $\lim_{i \to 0^+} H(i) = 0$ and from l'Hôpital rule and after some elementary calculations, $\lim_{i \to +\infty} H(i) = +\infty$. In order to analyze this dynamic system, we first show existence and uniqueness of a steady state and then local stability.

3. Characterization of steady state solutions

The above optimization problem differs from the standard optimal growth model with adjustment costs because of the presence of maintenance costs and the underutilization of capital. Therefore, before discussing the properties of the optimal steady state, we establish its existence and uniqueness in Proposition 1. We also define a golden rule for this economy, that we call *delta golden rule*, and we study the dynamic efficiency of the unique steady state solution and its relation with those corresponding to the benchmark neoclassical framework.

3.1 Existence and uniqueness of steady state solutions

The following proposition establishes sufficient conditions for the existence and uniqueness of a steady state equilibrium.

PROPOSITION 1. *Under the following limit conditions:*

i) $|\lim_{u \to 0+} m_\delta(\delta, u)| < 1 + \phi'(n + \delta)$, *for all* $\delta > 0$

ii) $|\lim_{u\to 1^-} m_\delta(\delta,u)| > 1+\phi'(n+\delta)$, *for all* $\delta > 0$

An interior steady state exists and it is unique.

PROOF. From equation [11] $H(i) = \beta$ at steady state. Given that $H(0) = 0$, $H'(i) > 0, \forall i > 0$, and $\lim_{i\to+\infty} H(i) = +\infty$, we may easily conclude that for any $\beta > 0$ there is only one positive value for the investment rate, $i = H^{-1}(\beta) > 0$. Then, from [12] the steady state depreciation rate $\delta = H^{-1}(\beta) - n$. Given that $H(i) < i$ and $H'(i) > 0, H^{-1}(\beta) > \beta > n$, which implies $\delta > 0$.

We combine [4] and [5] to obtain $1+\phi'(i) = -m_\delta(\delta,u)$. From $m_{\delta u}(\delta,u) < 0$, the right hand side of this equation is increasing in u. Given i and δ, conditions i) and ii) are sufficient for the existence of a unique solution for $u \in]0,1[$. From [6], $f'(ku) = m_u(\delta,u)$. Given that δ and u are interior at steady state, $m_u(\delta,u)$ is a strictly positive finite number. From the Inada conditions imposed on function $f(.)$, there exists one and only one interior steady state value for k. A steady state value for c can be obtained from [9], and the existence and uniqueness of a finite solution for it can be easily verified, given our assumptions on functions $f(.), \phi(.)$ and $m(.)$. To prove positivity, let us combine [9] with the other optimal conditions and the assumption of linear homogeneity of $m(.)$ and get

$$c = f(ku) - f'(ku)ku + k(1+\phi'(i))(H(i)-n) > 0$$

Given our assumptions on function $f(.)$, the net of the first two term on the right hand side is positive. At steady state, $H(i) = \beta > n$, which implies that the last term is also positive.

From [4], an interior solution for μ exists and is unique.

3.2 The (modified) delta golden rule

In our framework, the feasibility constraint at steady state can be written as

$$c = f(ku) - k\left[\delta + n + \phi(\delta+n) + m(\delta,u)\right].$$

Of course, the degree of capital utilization and the depreciation rate are not exogenously given. In order to make intertemporal efficiency comparisons we define a *delta golden rule.*

DEFINITION 3. *The* delta golden rule *is the value of* k *consistent with the maximization of steady state consumption with respect to* k, δ *and*

u, i.e., the solution of the following system:

$$f'(k\,u)\,u = \delta + n + \phi(\delta + n) + m(\delta, u) \qquad [13]$$

$$f'(k\,u) = m_u(\delta, u)$$

$$1 + \phi'(\delta + n) = -m_\delta(\delta, u).$$

Notice that the last two equations in Definition 3 are equal to equations [6] and [4]/[5], respectively. In the following proposition, we show that an interior delta golden rule exists and is unique.

PROPOSITION 2. *Under conditions i) and ii) of Proposition 1 together with $n > 0$, there is a unique delta golden rule.*

PROOF. Combining the three equations in Definition 3, we get

$$H(\delta + n) = n.$$

Provided that $n > 0$, since $H'(i) > 0, H(n) < n$, and $lim_{i \to +\infty} H(i) = +\infty$, there exists one and only one $\delta > 0$ satisfying Definition 3.

For a finite δ, the left hand side of the last equation in Definition 3 is constant. The right hand side is increasing in u, since $m_{\delta u}(\delta, u) < 0$. Then, conditions i) and ii), are sufficient for the existence of a unique solution for $u \in]0, 1[$.

Given a finite $\delta > 0$ and $u \in]0, 1[$, the right hand side of the first equation in Definition 3 is finite. From $f''(.) < 0$ and the Inada conditions, one and only one interior solution for k does exist.

For obvious reasons, any steady state value for the per capita capital stock that exceeds the delta golden rule value, denoted k_g, is dynamically inefficient irrespective of the corresponding values for δ and u. Of course, the steady state of our model economy, denoted k^*, is optimal and verifies that $k^* \leq k_g$. For further comparative analysis it is convenient to express equation [11] at steady state equilibrium values

$$f'(k^*\,u^*)u^* = \delta^* + \beta + \phi(\delta^* + n) + m(\delta^*, u^*) + (\beta - n)\phi'(\delta^* + n), \quad [14]$$

We call this equation the *modified delta golden rule*, to distinguish it from the *modified golden rule* of the Ramsey-Cass-Koopmans model. We should note from [14] that determination of the optimal capital stock requires the simultaneous determination of all control variables. Consequently, there are important sources of variation in steady-state

solutions related to changes in the parameters of the maintenance and adjustment costs technologies. This is an important implication of our model specification that goes beyond the somewhat counterfactual differences in preferences, population growth and technical progress the standard model needs to account for income differences across countries.

The following proposition shows the relation between the modified delta golden rule and the delta golden rule for some key variables.

PROPOSITION 3. *The following relations between the modified delta golden rule and the delta golden rule must hold:* $\delta^* > \delta_g$, $i^* > i_g$, $u^* > u_g$, $k^* < k_g$ *and* $y^* < y_g$.

PROOF. The golden rule implies $\delta_g = H^{-1}(n) - n$, while at steady state $\delta^* = H^{-1}(\beta) - n$. Since $\beta > n$ and $H(.)$ is monotonically increasing, $H^{-1}\beta > H^{-1}(n)$. Thus, $\delta^* > \delta_g$.

Moreover, given $i_g = \delta_g + n = H^{-1}(n)$ and $i^* = \delta^* + n = H^{-1}\beta$, we also conclude that $i^* > i_g$.

For both the golden and the modified golden rule, $1 + \phi'(\delta + n) = -m_\delta(\delta, u)$. Given that $\phi'' > 0$ and $\delta^* > \delta_g$, then $1 + \phi'(\delta_g + n) < 1 + \phi'(\delta^* + n)$, implying that $m_\delta(\delta_g, u_g) > m_\delta(\delta^*, u^*)$. Now, given $m_{\delta\delta} > 0, m_\delta(\delta^*, u^*) > m_\delta(\delta_g, u^*)$. The previous statements imply that $m_\delta(\delta_g, u_g) > m_\delta(\delta^*, u^*) > m_\delta(\delta_g, u^*)$, and the comparison between the two extreme terms, given $m_{\delta u} < 0$, says that $u^* > u_g$.

By definition, the golden rule implies $c_g \geq c^*$. Using the aggregate resource constraint, we get

$$f(k_g u_g) - (\delta_g + n + \phi(\delta_g + n) + m(\delta_g, u_g))k_g \geq$$
$$f(k^* u^*) - (\delta^* + n + \phi(\delta^* + n) + m(\delta^*, u^*))k^*.$$

Then, using equation [13] and [14], and rearranging terms we obtain the following inequality:

$$f(k_g u_g) - f'(k_g u_g)k_g u_g \geq$$
$$f(k^* u^*) - f'(k^* u^*)k^* u^* + (\beta - n)[1 + \phi'(\delta^* + n)]k^* >$$
$$f(k^* u^*) - f'(k^* u^*)k^* u^*.$$

So, given the strict concavity of the production function we may easly conclude that $k_g u_g > k^* u^*$. Then given $u_g < u^*$, it becomes obvious that $k^* < k_g$.

From the previous result, $y^* < y_g$.

Thus, a higher productivity of physical capital in the long-run is associated with long-run levels of the depreciation rate, the investment rate and the utilization rate that are above those of the golden-rule solution. This result is standard in the optimal growth literature. Less immediate results show up through comparison with the Ramsey-Cass-Koopmans and the Abel-Blanchard models.

Indeed, the equilibria of the Ramsey-Cass-Koopmans model (say, k_R) and that of Abel and Blanchard (say, k_A) are characterized by

$$f'(k_R) = \delta + \beta \qquad [15]$$

and

$$f'(k_A) = \delta + \beta + \hat{\phi}(\delta + n) + (\beta - n)\hat{\phi}'(\delta + n) \qquad [16]$$

respectively, where the utilization rate is supposed to be equal to one and the depreciation rate is an exogenous parameter. Under standard assumptions on the adjustment cost function, $\hat{\phi}(.)$ should be positive, which implies that $k_A \leq k_R$. In Proposition 4, we compare the Abel and Blanchard steady state equilibrium with our delta and modified delta golden rules. In order to do this comparison, we assume that $\delta = \delta^* + m^*$, where $m^* = m(\delta^*, u^*)$, and $\hat{\phi}(i) \equiv \phi(i - m^*)$. The rationale for these assumptions is the following. In our model, maintenance and repair are considered separate economic activities, but in Abel and Blanchard, investment includes them. Consequently, in our model depreciation is net of maintenance, but not in Abel and Blanchard. For the same reason, we must renormalize the investment function: it depends on gross investment in Abel and Blanchard and on investment net of maintenance in our model.

PROPOSITION 4. $k^* < k_A$ *if* $\Theta(k) \equiv -\frac{kf''(k)}{f'(k)} < 1$.

PROOF. Since $\delta = \delta^* + m(\delta^*, u^*), \hat{\phi}(x) = \phi(x - m(\delta^*, u^*))$ and $u^* < 1$, from [14] and [16] we get $f'(k^*u^*) > f'(k^*u^*)u^* = f'(k_A)$. Consequently, $k^*u^* < k_A$ and

$$\frac{k_A}{k^*} = \frac{f'(k_A)k_A}{f'(k^*u^*)k^*u^*}.$$

Let us define $G(k) \equiv f'(k)k$. We can easily prove that, for $k > 0$, $G'(k) > 0 \Leftrightarrow \Theta(k) < 1$. Then,

$$\frac{f'(k_A)k_A}{f'(k^*u^*)k^*u^*} > 1 \qquad if \qquad \Theta(k) < 1,$$

which completes the proof.

$\Theta(k)$ represents the curvature of the production technology $f(k)$. For a CES production function with elasticity of substitution larger than one, this assumption holds for any $k > 0$. This result can also hold for a CES production function with gross complementarity, provided that k_A is not too large, since $\lim_{k\to 0+} \Theta(k) = 0$ independently of the elasticity of substitution.

The intuition behind the result in Proposition 4 is straightforward. A non-optimal depreciation policy, that ignores variable maintenance costs and utilization, leads the economy to an excess of installed capacity. However, this excess of capacity is not necessarily dynamically inefficient.

4. Dynamic analysis of optimal trajectories

In Proposition 5 we prove local stability of the unique interior steady state equilibrium.

PROPOSITION 5. *The unique interior steady state equilibrium of the dynamic system [11] and [12] is locally saddle-path stable.*

PROOF. A first order Taylor expansion of [11] and [12] may be written in matrix form as

$$\begin{pmatrix} \dot{k} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} \Pi_k(k^*, \mu^*) & \Pi_\mu(k^*, \mu^*) \\ \Gamma_k(k^*, \mu^*) & \Gamma_\mu(k^*, \mu^*) \end{pmatrix} \begin{pmatrix} k - k^* \\ \mu - \mu^* \end{pmatrix},$$

where x^* denotes the steady state value of x. As it is shown in the Appendix, the coefficients of the Jacobian matrix J^* are:

$$\begin{aligned} \Pi_k(k^*, \mu^*) &= k^*(i_k^* - \delta_k^*) > 0 \\ \Pi_\mu(k^*, \mu^*) &= k^*(i_\mu^* - \delta_\mu^*) > 0 \\ \Gamma_k(k^*, \mu^*) &= -\mu^* H'(i^*) i_k^* > 0 \\ \Gamma_\mu(k^*, \mu^*) &= -\mu^* H'(i^*) i_\mu^* < 0 \end{aligned}$$

Following Kurz (1968), the trace of the Jacobian matrix must be:

$$\text{trace } J* = \gamma_1 + \gamma_2 = k^*[i_k^* - \delta_k^*] - \mu^* H'(i^*) i_\mu^* = \beta - n > 0,$$

where γ_1 and γ_2 are the eigenvalues. On the other hand, the determinant of the Jacobian matrix is:

$$\det J^* = \gamma_1 \cdot \gamma_2 = \mu^* H'(i^*) k^* [i_\mu^* \delta_k^* - i_k^* \delta_\mu^*] < 0 \qquad [17]$$

Consequently, the two eigenvalues are, respectively, $\gamma_1 = \frac{\beta-n}{2} + \sqrt{\left[\frac{\beta-n}{2}\right]^2 - |\det J^*|} > 0$ and $\gamma_2 = \frac{\beta-n}{2} + \sqrt{\left[\frac{\beta-n}{2}\right]^2 - |\det J^*|} < 0$. These features of the Jacobian matrix mean that the system has a saddle point dynamical structure. So, given the initial condition for the predetermined variable k_0 and the transversality condition [10], the system places on the stable arm and then converges to the steady state equilibrium. Otherwise the system explodes.

In particular, given $k_0 < k^*$ convergence implies that $k(t)$ increases but $\mu(t)$ decreases because $\mu(0) > \mu^*$. The speed of convergence, measured as the absolute value of the negative eigenvalue, is given by $\upsilon = -\gamma_2$, with $\frac{\partial \upsilon}{\partial |\det J^*|} > 0$.

It seems difficult to state general conditions characterizing the speed of convergence in our model. To further examine the dynamic properties of the convergence path we resort to numerical computations. First, we investigate the impact of our technological assumptions in the speed of convergence of the neoclassical growth model. The following specific functional forms are used throughout: $U(c) = \ln c$, $f(ku) = (ku)^\alpha$, $\phi(i) = (b/2)i^2$ and $m(\delta, u) = du^2/\delta$.

TABLE 1
Effect of parameters b and α on steady-state values and the rates of convergence

b	α	δ^*	u^*	k^*	k_g	k_A	$\nu_A(\%)$	$\nu(\%)$
10	0.16	0.048	0.85	1.04	1.36	1.07	9.72	3.84
6.5	0.40	0.059	1.00	4.21	5.43	4.22	8.23	3.40
10	0.40	0.048	0.85	4.58	6.07	5.09	6.50	2.80
24	0.40	0.033	0.66	4.89	6.97	6.47	4.12	2.00
140	0.40	0.018	0.55	2.91	5.20	4.34	2.00	1.10
10	0.75	0.048	0.85	381.80	541.60	615.00	2.50	1.45

We consider our model, together with the Abel and Blanchard model under the corresponding interpretations of the depreciation rate and the investment rate as discussed above. We fix parameters $\beta = 0.01$, $n = 0.0075$ and $d = 0.005$, the other parameters varying as specified in Table 1.[4] It is worth pointing out that parameter d of the maintenance costs function affects the steady-state values u^*, k^* and k_g but not the

[4]The benchmark value of the adjustment cost coefficient, $b = 10$, is chosen to get a plausible value for Tobin's q (1.5). For the benchmark value of the capital elasticity, $\alpha = 0.40$, we consider variations in parameter b such that $u = 1$, $\nu_A = 2\%$ and $\nu = 2\%$, respectively.

rate of convergence. In this table, ν_A is the speed of convergence of the Abel and Blanchard model.

The main conclusion that can be drawn from Table 1 is that the rate of convergence in our model is substantially reduced compared with the standard neoclassical growth model with adjustment costs. Indeed, the decrease in the speed of convergence we obtain assigns more importance to the transitional dynamics of our model. This result is in line with related literature that shows the importance of variable utilization rates of capital in shaping the saddle path and the convergence rate. For instance, Rumbos and Auernheimer (1997) quantitatively compare the rate of convergence of small open model economies with fixed and variable utilization rates. They find slower convergence under variable utilization, the absolute differences going up to 18 per cent. Here we have a lot more intratemporal substitution between investment and maintenance through a variable utilization rate and an endogenous depreciation rate; so much so that convergence is at least twice as fast in the standard neoclassical model with adjustment costs. Furthermore, for our baseline economy with $b = 10$ and $\alpha = 0.40$ we obtain a rate of convergence which is more in conformity with those reported in some empirical studies (Barro and Sala-i-Martin, 1992, report annual rates of convergence of the order of 2 per cent).

Finally, Table 2 summarizes the dynamic behavior of the variables in our model along the capital accumulation convergence path for alternative values of b and α. In all our numerical experiments consumption and investment react as in the standard neoclassical model.

TABLE 2
Effect of parameters b and α on the dynamic behavior of variables along the convergence path

b	α	du/dk	di/dk	$d\delta/dk$	dc/dk	dm/dk
10	0.16	-0.6159	-0.0621	-0.0253	0.0637	-0.0694
6.5	0.40	-0.1752	-0.0163	-0.0082	0.0523	-0.0179
10	0.40	-0.1320	-0.0119	-0.0056	0.0484	-0.0145
24	0.40	-0.0873	-0.0071	-0.0030	0.0444	-0.0115
140	0.40	-0.1089	-0.0057	-0.0020	0.0604	-0.0237
10	0.75	-0.0012	-0.0001	-0.0001	0.0265	-0.0001

As the initial stock of capital is below its steady state value, the optimal initial reaction is to accumulate, maintain and utilize capital at higher rates than in steady state. For this reason, in the adjustment

process all these rates are decreasing. However, the depreciation rate does not initially decrease, but increase. This depends crucially on the negative sign of the cross derivative in the maintenance function: from [5], utilization and depreciation move in the same direction.

5. Conclusions

In this paper we introduce maintenance costs, endogenous depreciation and capital utilization rates in an otherwise optimal growth model with investment adjustment costs. Our model specification generalizes that of Abel and Blanchard (1983) and provides a theoretical framework for the analysis of comparative dynamics in a class of general equilibrium models with capital underutilization and maintenance activities.

The social optimum is characterized by an endogenous simultaneous determination of the investment rate, the depreciation rate and the utilization rate. This circumstance has some relevant implications. First, in steady state capital is optimally underutilized and maintenance activities are optimally undertaken. Second, we define a delta golden rule and we show that the steady state equilibrium of our economy verifies a modified delta golden rule, which is dynamically efficient. Finally, we show that the unique steady state equilibrium is locally stable.

But we have taken only one necessary step in characterizing the equilibrium dynamics of a growth model with endogenous depreciation, capital underutilization and spending on maintenance and repair. Further work, in line with Licandro and Puch (2000), is needed to definitively establish the extent to which including these features in aggregate models will change the answers to quantitative questions.

Appendix. Control functions and their partial derivatives at steady state

Taking equations [4]-[6], [9], and the production function in intensive form $y = f(ku)$, we can implicitly define the following optimal control functions: $u = u(k, \mu)$, $i = i(k, \mu)$, $\delta = \delta(k, \mu)$, $c = c(k, \mu)$, and $y = y(k, \mu)$. Via the implicit function theorem we can identify the corresponding partial effects, evaluated at the steady state where $H(i) + \delta - i = \beta - n > 0$:

$$u_k = \frac{f''u\{[U']^2\phi''m_{\delta\delta} - U''k[m_\delta]^2U'm_{\delta\delta} - U'\phi''U''k[m_\delta]^2\}}{\Delta} -$$

$$
\begin{aligned}
& - \frac{[\beta - n]U'\phi'' m_{u\delta}[m_\delta]^2 U''}{\Delta} < 0 \\
u_\mu &= \frac{U'\phi'' m_{u\delta}}{\Delta} < 0 \\
i_k &= \frac{U'' k[m_\delta]^2 U' f'' m_{\delta\delta}[\beta - \delta - n]}{\Delta} < 0 \\
i_\mu &= -\frac{U' f'' k m_{\delta\delta}}{\Delta} > 0 \\
\delta_k &= -\frac{U' m_{u\delta} f'' u\{U'\phi'' - U'' k[m_\delta]^2\}}{\Delta} - \\
& - \frac{[m_\delta]^2 U''[\beta - n]U'\phi''[m_{uu} - f'' k]}{\Delta} < 0 \\
\delta_\mu &= -\frac{U'\phi''[m_{uu} - f'' k]}{\Delta} < 0 \\
c_k &= \frac{m_\delta[\beta - \delta - n] f'' k[U']^2 m_{\delta\delta}\phi''}{\Delta} < 0 \\
c_\mu &= -\frac{k m_\delta\{U' f'' k m_{\delta\delta} - U'\phi''[m_{uu} - f'' k]\}}{\Delta} < 0 \\
y_k &= \frac{f' u\Delta + f' k f'' u\{[U']^2\phi'' m_{\delta\delta}}{\Delta} - \\
& - \frac{U'' k[m_\delta]^2 U' m_{\delta\delta} - U'\phi'' U'' k[m_\delta]^2\}}{\Delta} - \\
& - \frac{f' k[\beta - n]U'\phi'' m_{u\delta}[m_\delta]^2 U''}{\Delta} > 0 \\
y_\mu &= \frac{f' k U'\phi'' m_{u\delta}}{\Delta} < 0
\end{aligned}
$$

where $\Delta = -U'\phi'' U'' k[m_\delta]^2[m_{uu} - f'' k] - f'' k U' m_{\delta\delta}[U'\phi'' - U'' k[m_\delta]^2] > 0$. Finally, after some elementary manipulations, we obtain that $i_k - \delta_k > 0$.

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Resumen

En este artículo se analiza la dinámica de un modelo de crecimiento que incorpora depreciación endógena, utilización variable del capital y gastos de mantenimiento. La actividad de mantenimiento reduce la depreciación del capital, la inversión está sujeta a costes y el grado de utilización del capital afecta a los costes de mantenimiento. En este marco se establecen condiciones suficientes para la existencia y unicidad del equilibrio de largo plazo. Además, se define una "delta golden rule" consistente con la economía considerada y se analiza la eficiencia dinámica de esta economía. Por último se muestra que el estado estacionario es localmente estable

Palabras clave: Depreciación, utilización del capital, crecimiento óptimo.

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